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Common Eigenvectors of Four Particles' Compatible Observables

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Abstract We construct eight operators for a four-particle system, namely one center-of-mass coordinate operator, three relative coordinate operators, one total momentum operator and three mass-weighted relative momentum operators, and give common eigenvectors of four compatible observables $\{\sum_{i=1}^{4}\hat{p}_i, \hat{x}_1-\hat{x}_2, \hat{x}_2-\hat{x}_3, \hat{x}_3-\hat{x}_4\}$, which are composed of four particles' coordinate $\hat{x}_i$ and momentum $\hat{p}_i$. By compatible we mean such observables can be simultaneously determined. Using the technique of integration within an ordered product (IWOP) of operators, we prove that the common eigenvectors are complete and orthonormal, and hereby qualified for making up a representation.

Keywords IWOP technique · Eigenvectors · Four particles

1 Introduction

Fan and Klauder [1] had constructed the explicit form of the common eigenvectors of two particles' total momentum $\hat{p}_1+\hat{p}_2$ and relative position $\hat{x}_1-\hat{x}_2$ in the two-mode Fock space. Fan and Zhang [2] had given The common eigenkets of three compatible observables $\{\hat{p}_1+\hat{p}_2+\hat{p}_3, (\mu_2\hat{x}_2+\mu_3\hat{x}_3)/(\mu_2+\mu_3)-\hat{x}_1, \hat{x}_3-\hat{x}_2\}$. The authors of [3–12] had studied the representation theory and their applications widely. These work have greatly developed representation theory. But to our knowledge, the common eigenvectors of four compatible observables for a four-particle system haven't been constructed. The purpose of this work is to construct common eigenvectors for a set of four-particle compatible observables $\{\hat{P}_1, \hat{x}_1-\hat{x}_2, \hat{x}_2-\hat{x}_3, \hat{x}_3-\hat{x}_4\}$, where $\hat{P}_1=\hat{p}_1+\hat{p}_2+\hat{p}_3+\hat{p}_4$ is the total momentum.

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We construct eight operators for a four-particle system as follows

$$\begin{cases} \hat{P}_1 = \sum_{i=1}^4 \hat{p}_i, \\ \hat{P}_2 = \left(\sum_{i=2}^4 \mu_i\right)\hat{p}_1 - \mu_1\left(\sum_{i=2}^4 \hat{p}_i\right), \\ \hat{P}_3 = \left(\sum_{i=3}^4 \mu_i\right)(\hat{p}_1 + \hat{p}_2) - (\mu_1 + \mu_2)\left(\sum_{i=3}^4 \hat{p}_i\right), \\ \hat{P}_4 = \mu_4\left(\sum_{i=1}^3 \hat{p}_i\right) - \left(\sum_{i=1}^3 \mu_i\right)\hat{p}_4, \end{cases} \qquad \begin{cases} \hat{X}_1 = \sum_{i=1}^4 (\mu_i \hat{x}_i), \\ \hat{X}_2 = \hat{x}_1 - \hat{x}_2, \\ \hat{X}_3 = \hat{x}_2 - \hat{x}_3, \\ \hat{X}_4 = \hat{x}_3 - \hat{x}_4, \end{cases}$$

where $\mu_i = m_i/M$, $i = 1, 2, 3, 4$, $M = m_1 + m_2 + m_3 + m_4$.

It is easy to see that $\hat{P}_1$, $\hat{P}_2$, $\hat{P}_3$, $\hat{P}_4$ are linear independent, and $\hat{X}_1$, $\hat{X}_2$, $\hat{X}_3$, $\hat{X}_4$ are linear independent, which satisfy the following relations

$$[\hat{X}_i, \hat{P}_j] = \mathrm{i}\hbar\delta_{ij}, \quad i, j = 1, 2, 3, 4. \tag{1}$$

We are challenged to search for the common eigenvectors $|p, \chi_2, \chi_3, \chi_4\rangle$ of these four compatible observables $\{\hat{P}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4\}$. By virtue of the technique of integration within an ordered product of operators (IWOP), we show that $|p, \chi_2, \chi_3, \chi_4\rangle$ makes a complete representation. Thus they can be used for solving some new four-body dynamic problems.

2 Common Eigenvectors of the Four Compatible Observables

We begin with introducing the four-mode Fock space spanned by $|n_1, n_2, n_3, n_4\rangle = (\hat{a}_1^{+n_1}\hat{a}_2^{+n_2}\hat{a}_3^{+n_3}\hat{a}_4^{+n_4}/\sqrt{n_1!n_2!n_3!n_4!})|0000\rangle$, where $[\hat{a}_i, \hat{a}_j^+] = \delta_{ij}$, $|0000\rangle$ is the ground state, and $\hat{a}_i$ and $\hat{a}_i^+$ are related to $\hat{x}_i$ and $\hat{p}_i$ by

$$\hat{x}_i = \frac{\hat{a}_i + \hat{a}_i^+}{\sqrt{2}}, \qquad \hat{p}_i = \frac{\hat{a}_i - \hat{a}_i^+}{\mathrm{i}\sqrt{2}}, \quad [\hat{a}_i, \hat{a}_j^+] = \delta_{ij}. \tag{2}$$

We shall prove that the common eigenvectors of $\{\sum_{i=1}^4 \hat{p}_i, \hat{x}_1 - \hat{x}_2, \hat{x}_2 - \hat{x}_3, \hat{x}_3 - \hat{x}_4\}$, denoted by $|p, \chi_2, \chi_3, \chi_4\rangle$, are given in four-mode Fock space by

$$\begin{aligned} |p, \chi_2, \chi_3, \chi_4\rangle = \frac{\pi^{-1}}{2}\exp\bigg\{ & -\frac{1}{8}p^2 - \frac{3}{8}\chi_2^2 - \frac{1}{2}\chi_3^2 - \frac{3}{8}\chi_4^2 - \frac{1}{2}\chi_2\chi_3 - \frac{1}{4}\chi_2\chi_4 - \frac{1}{2}\chi_3\chi_4 \\ & + \frac{\sqrt{2}}{4}(\mathrm{i}p + 3\chi_2 + 2\chi_3 + \chi_4)\hat{a}_1^+ + \frac{\sqrt{2}}{4}(\mathrm{i}p - \chi_2 + 2\chi_3 + \chi_4)\hat{a}_2^+ \\ & + \frac{\sqrt{2}}{4}(\mathrm{i}p - \chi_2 - 2\chi_3 + \chi_4)\hat{a}_3^+ + \frac{\sqrt{2}}{4}(\mathrm{i}p - \chi_2 - 2\chi_3 - 3\chi_4)\hat{a}_4^+ \\ & - \mathrm{i}p\left[\left(\frac{1}{4} - \mu_1\right)\chi_2 + \left(\frac{1}{2} - \mu_1 - \mu_2\right)\chi_3 + \left(\frac{3}{4} - \mu_1 - \mu_2 - \mu_3\right)\chi_4\right] \\ & - \frac{1}{4}(\hat{a}_1^{+2} + \hat{a}_2^{+2} + \hat{a}_3^{+2} + \hat{a}_4^{+2}) \\ & + \frac{1}{2}(\hat{a}_1^+\hat{a}_2^+ + \hat{a}_1^+\hat{a}_3^+ + \hat{a}_1^+\hat{a}_4^+ + \hat{a}_2^+\hat{a}_3^+ + \hat{a}_2^+\hat{a}_4^+ + \hat{a}_3^+\hat{a}_4^+)\bigg\}|0000\rangle. \end{aligned} \tag{3}$$

In fact, acting $\hat{a}_i$ on $|p,\chi_2,\chi_3,\chi_4\rangle$ we have

$$\hat{a}_1|p,\chi_2,\chi_3,\chi_4\rangle = \left[\frac{\sqrt{2}}{4}(\mathrm{i}p+3\chi_2+2\chi_3+\chi_4)-\frac{\hat{a}_1^+}{2}+\frac{\hat{a}_2^+ +\hat{a}_3^+ +\hat{a}_4^+}{2}\right]|p,\chi_2,\chi_3,\chi_4\rangle\,, \tag{4}$$

$$\hat{a}_2|p,\chi_2,\chi_3,\chi_4\rangle = \left[\frac{\sqrt{2}}{4}(\mathrm{i}p-\chi_2+2\chi_3+\chi_4)-\frac{\hat{a}_2^+}{2}+\frac{\hat{a}_1^+ +\hat{a}_3^+ +\hat{a}_4^+}{2}\right]|p,\chi_2,\chi_3,\chi_4\rangle\,, \tag{5}$$

$$\hat{a}_3|p,\chi_2,\chi_3,\chi_4\rangle = \left[\frac{\sqrt{2}}{4}(\mathrm{i}p-\chi_2-2\chi_3+\chi_4)-\frac{\hat{a}_3^+}{2}+\frac{\hat{a}_1^+ +\hat{a}_2^+ +\hat{a}_4^+}{2}\right]|p,\chi_2,\chi_3,\chi_4\rangle\,, \tag{6}$$

$$\hat{a}_4|p,\chi_2,\chi_3,\chi_4\rangle = \left[\frac{\sqrt{2}}{4}(\mathrm{i}p-\chi_2-2\chi_3-3\chi_4)-\frac{\hat{a}_4^+}{2}+\frac{\hat{a}_1^+ +\hat{a}_2^+ +\hat{a}_3^+}{2}\right]|p,\chi_2,\chi_3,\chi_4\rangle\,. \tag{7}$$

The sum of (4)–(7) leads to

$$\hat{P}_1|p,\chi_2,\chi_3,\chi_4\rangle = \left(\sum_{i=1}^{4}\hat{p}_i\right)|p,\chi_2,\chi_3,\chi_4\rangle = p\,|p,\chi_2,\chi_3,\chi_4\rangle\,, \tag{8}$$

$$\hat{X}_2|p,\chi_2,\chi_3,\chi_4\rangle = (\hat{x}_1-\hat{x}_2)|p,\chi_2,\chi_3,\chi_4\rangle = \chi_2|p,\chi_2,\chi_3,\chi_4\rangle\,, \tag{9}$$

$$\hat{X}_3|p,\chi_2,\chi_3,\chi_4\rangle = (\hat{x}_2-\hat{x}_3)|p,\chi_2,\chi_3,\chi_4\rangle = \chi_3|p,\chi_2,\chi_3,\chi_4\rangle\,, \tag{10}$$

$$\hat{X}_4|p,\chi_2,\chi_3,\chi_4\rangle = (\hat{x}_3-\hat{x}_4)|p,\chi_2,\chi_3,\chi_4\rangle = \chi_4|p,\chi_2,\chi_3,\chi_4\rangle\,. \tag{11}$$

Thus we know $|p,\chi_2,\chi_3,\chi_4\rangle$ is the eigenvectors we required.

3 The Completeness Relation of $|p,\chi_2,\chi_3,\chi_4\rangle$

We now examine if $|p,\chi_2,\chi_3,\chi_4\rangle$ satisfies a completeness relation. By virtue of IWOP technique and the normal ordering form of the four-mode vacuum projector

$$|0000\rangle\langle 0000| =: \exp\{-\hat{a}_1^+\hat{a}_1-\hat{a}_2^+\hat{a}_2-\hat{a}_3^+\hat{a}_3-\hat{a}_4^+\hat{a}_4\}: \tag{12}$$

We can easily perform the following integration:

$$\begin{aligned}
&\iiiint_{-\infty}^{\infty} dp\,d\chi_2 d\chi_3 d\chi_4\,|p,\chi_2,\chi_3,\chi_4\rangle\langle p,\chi_2,\chi_3,\chi_4| \\
&= \frac{\pi^{-2}}{4}\iiiint_{-\infty}^{\infty} dp\,d\chi_2 d\chi_3 d\chi_4 : \exp\left\{-\frac{1}{4}p^2-\frac{3}{4}\chi_2^2-\chi_3^2-\frac{3}{4}\chi_4^2-\chi_2\chi_3\right. \\
&\quad -\frac{1}{2}\chi_2\chi_4-\chi_3\chi_4+\mathrm{i}\frac{\sqrt{2}}{4}[\hat{a}_1^+-\hat{a}_1+\hat{a}_2^+-\hat{a}_2+\hat{a}_3^+-\hat{a}_3+\hat{a}_4^+-\hat{a}_4]p \\
&\quad +\frac{\sqrt{2}}{4}[3(\hat{a}_1^+ +\hat{a}_1)-(\hat{a}_2^+ +\hat{a}_2)-(\hat{a}_3^+ +\hat{a}_3)-(\hat{a}_4^+ +\hat{a}_4)]\chi_2
\end{aligned}$$

$$+\frac{\sqrt{2}}{4}[2(\hat{a}_1^+ + \hat{a}_1) + 2(\hat{a}_2^+ + \hat{a}_2) - 2(\hat{a}_3^+ + \hat{a}_3) - 2(\hat{a}_4^+ + \hat{a}_4)]\chi_3$$
$$+\frac{\sqrt{2}}{4}[(\hat{a}_1^+ + \hat{a}_1) + (\hat{a}_2^+ + \hat{a}_2) + (\hat{a}_3^+ + \hat{a}_3) - 3(\hat{a}_4^+ + \hat{a}_4)]\chi_4$$
$$-\frac{\hat{a}_1^{+^2} + \hat{a}_1^2 + \hat{a}_2^{+^2} + \hat{a}_2^2 + \hat{a}_3^{+^2} + \hat{a}_3^2 + \hat{a}_4^{+^2} + \hat{a}_4^2}{4}$$
$$+\frac{\hat{a}_1^+\hat{a}_2^+ + \hat{a}_1^+\hat{a}_3^+ + \hat{a}_1^+\hat{a}_4^+ + \hat{a}_2^+\hat{a}_3^+ + \hat{a}_2^+\hat{a}_4^+ + \hat{a}_3^+\hat{a}_4^+}{2}$$
$$+\frac{\hat{a}_1\hat{a}_2 + \hat{a}_1\hat{a}_3 + \hat{a}_1\hat{a}_4 + \hat{a}_2\hat{a}_3 + \hat{a}_2\hat{a}_4 + \hat{a}_3\hat{a}_4}{2} - \sum_{i=1}^{4}(\hat{a}_i^+ a_i)\Bigg\} := 1. \tag{13}$$

Using (8)–(11) we can also calculate

$$\begin{aligned}\langle p', \chi_2', \chi_3', \chi_4'|\hat{P}_1|p, \chi_2, \chi_3, \chi_4\rangle &= p'\left\langle p', \chi_2', \chi_3', \chi_4' \mid p, \chi_2, \chi_3, \chi_4\right\rangle \\ &= p\left\langle p', \chi_2', \chi_3', \chi_4' \mid p, \chi_2, \chi_3, \chi_4\right\rangle,\end{aligned} \tag{14}$$

$$\begin{aligned}\langle p', \chi_2', \chi_3', \chi_4'|\hat{X}_2|p, \chi_2, \chi_3, \chi_4\rangle &= \chi_2'\left\langle p', \chi_2', \chi_3', \chi_4' \mid p, \chi_2, \chi_3, \chi_4\right\rangle \\ &= \chi_2\left\langle p', \chi_2', \chi_3', \chi_4' \mid p, \chi_2, \chi_3, \chi_4\right\rangle,\end{aligned} \tag{15}$$

$$\begin{aligned}\langle p', \chi_2', \chi_3', \chi_4'|\hat{X}_3|p, \chi_2, \chi_3, \chi_4\rangle &= \chi_3'\left\langle p', \chi_2', \chi_3', \chi_4' \mid p, \chi_2, \chi_3, \chi_4\right\rangle \\ &= \chi_3\left\langle p', \chi_2', \chi_3', \chi_4' \mid p, \chi_2, \chi_3, \chi_4\right\rangle,\end{aligned} \tag{16}$$

$$\begin{aligned}\langle p', \chi_2', \chi_3', \chi_4'|\hat{X}_4|p, \chi_2, \chi_3, \chi_4\rangle &= \chi_4'\left\langle p', \chi_2', \chi_3', \chi_4' \mid p, \chi_2, \chi_3, \chi_4\right\rangle \\ &= \chi_4\left\langle p', \chi_2', \chi_3', \chi_4' \mid p, \chi_2, \chi_3, \chi_4\right\rangle,\end{aligned} \tag{17}$$

which tell us that the overlap

$$\langle p', \chi_2', \chi_3', \chi'|p, \chi_2, \chi_3, \chi_4\rangle = \delta(p' - p)\delta(\chi_2' - \chi_2)\delta(\chi_3' - \chi_3)\delta(\chi_4' - \chi_4), \tag{18}$$

indicating that $|p, \chi_2, \chi_3, \chi_4\rangle$ are orthonormal.

From (13) and (18), we see that $|p, \chi_2, \chi_3, \chi_4\rangle$ are qualified for making up a representation.

4 The Representation of $\hat{P}_2$, $\hat{P}_3$ and $\hat{P}_4$ in $\langle p, \chi_2, \chi_3, \chi_4|$ Basis

From (2) and (3), it is easy to derive $\hat{P}_2$'s, $\hat{P}_3$'s and $\hat{P}_4$'s representation in $\langle p, \chi_2, \chi_3, \chi_4|$ basis. On the one hand, we have

$$\begin{aligned}\hat{P}_2\,|p, \chi_2, \chi_3, \chi_4\rangle &= \left[\left(\sum_{i=2}^{4}\mu_i\right)\hat{p}_1 - \mu_1\sum_{i=2}^{4}\hat{p}_i\right]|p, \chi_2, \chi_3, \chi_4\rangle \\ &= \frac{1}{\mathrm{i}\sqrt{2}}\left[\left(\sum_{i=2}^{4}\mu_i\right)(\hat{a}_1 - \hat{a}_1^+) - \mu_1\sum_{i=2}^{4}(\hat{a}_i - \hat{a}_i^+)\right]|p, \chi_2, \chi_3, \chi_4\rangle\end{aligned}$$

$$= \left[\left(\frac{1}{4} - \mu_1\right)p - \mathrm{i}\frac{3}{4}\chi_2 - \mathrm{i}\frac{1}{2}\chi_3 - \mathrm{i}\frac{1}{4}\chi_4 + \mathrm{i}\frac{3\sqrt{2}}{4}\hat{a}_1^+ - \mathrm{i}\frac{\sqrt{2}}{4}\hat{a}_2^+ - \mathrm{i}\frac{\sqrt{2}}{4}\hat{a}_3^+ - \mathrm{i}\frac{\sqrt{2}}{4}\hat{a}_4^+\right]|p, \chi_2, \chi_3, \chi_4\rangle, \tag{19}$$

$$\begin{aligned}
\hat{P}_3\,|p, \chi_2, \chi_3, \chi_4\rangle &= \left[\left(\sum_{i=3}^{4}\mu_i\right)(\hat{p}_1 + \hat{p}_2) - (\mu_1 + \mu_2)\sum_{i=3}^{4}\hat{p}_i\right]|p, \chi_2, \chi_3, \chi_4\rangle \\
&= \frac{1}{\mathrm{i}\sqrt{2}}\left[\left(\sum_{i=3}^{4}\mu_i\right)\sum_{j=1}^{2}(\hat{a}_i - \hat{a}_i^+) - (\mu_1 + \mu_2)\sum_{i=3}^{4}(\hat{a}_i - \hat{a}_i^+)\right] \\
&\quad \times |p, \chi_2, \chi_3, \chi_4\rangle \\
&= \left[\left(\frac{1}{2} - \mu_1 - \mu_2\right)p - \mathrm{i}\frac{1}{2}\chi_2 - \mathrm{i}\chi_3 - \mathrm{i}\frac{1}{2}\chi_4 + \mathrm{i}\frac{\sqrt{2}}{2}\hat{a}_1^+ + \mathrm{i}\frac{\sqrt{2}}{2}\hat{a}_2^+ \right. \\
&\quad \left. - \mathrm{i}\frac{\sqrt{2}}{2}\hat{a}_3^+ - \mathrm{i}\frac{\sqrt{2}}{2}\hat{a}_4^+\right]|p, \chi_2, \chi_3, \chi_4\rangle,
\end{aligned} \tag{20}$$

$$\begin{aligned}
\hat{P}_4\,|p, \chi_2, \chi_3, \chi_4\rangle &= \left[\mu_4\sum_{i=1}^{3}\hat{p}_i - \left(\sum_{i=1}^{3}\mu_i\right)\hat{p}_4\right]|p, \chi_2, \chi_3, \chi_4\rangle \\
&= \frac{1}{\mathrm{i}\sqrt{2}}\left[\mu_4\sum_{i=1}^{3}(\hat{a}_i - \hat{a}_i^+) - \left(\sum_{i=1}^{3}\mu_i\right)(\hat{a}_4 - \hat{a}_4^+)\right]|p, \chi_2, \chi_3, \chi_4\rangle \\
&= \left[\left(-\frac{3}{4} + \mu_4\right)p - \mathrm{i}\frac{1}{4}\chi_2 - \mathrm{i}\frac{1}{2}\chi_3 - \mathrm{i}\frac{3}{4}\chi_4 + \mathrm{i}\frac{\sqrt{2}}{4}\hat{a}_1^+ + \mathrm{i}\frac{\sqrt{2}}{4}\hat{a}_2^+ \right. \\
&\quad \left. + \mathrm{i}\frac{\sqrt{2}}{4}\hat{a}_3^+ - \mathrm{i}\frac{3\sqrt{2}}{4}\hat{a}_4^+\right]|p, \chi_2, \chi_3, \chi_4\rangle.
\end{aligned} \tag{21}$$

On the other hand, we have

$$\begin{aligned}
\mathrm{i}\frac{\partial}{\partial\chi_2}\,|p, \chi_2, \chi_3, \chi_4\rangle &= \left[\left(\frac{1}{4} - \mu_1\right)p - \mathrm{i}\frac{3}{4}\chi_2 - \mathrm{i}\frac{1}{2}\chi_3 - \mathrm{i}\frac{1}{4}\chi_4 + \mathrm{i}\frac{3\sqrt{2}}{4}\hat{a}_1^+ \right. \\
&\quad \left. - \mathrm{i}\frac{\sqrt{2}}{4}\hat{a}_2^+ - \mathrm{i}\frac{\sqrt{2}}{4}\hat{a}_3^+ - \mathrm{i}\frac{\sqrt{2}}{4}\hat{a}_4^+\right]|p, \chi_2, \chi_3, \chi_4\rangle,
\end{aligned} \tag{22}$$

$$\begin{aligned}
\mathrm{i}\frac{\partial}{\partial\chi_3}\,|p, \chi_2, \chi_3, \chi_4\rangle &= \left[\left(\frac{1}{2} - \mu_1 - \mu_2\right)p - \mathrm{i}\frac{1}{2}\chi_2 - \mathrm{i}\chi_3 - \mathrm{i}\frac{1}{2}\chi_4 + \mathrm{i}\frac{\sqrt{2}}{2}\hat{a}_1^+ \right. \\
&\quad \left. + \mathrm{i}\frac{\sqrt{2}}{2}\hat{a}_2^+ - \mathrm{i}\frac{\sqrt{2}}{2}\hat{a}_3^+ - \mathrm{i}\frac{\sqrt{2}}{2}\hat{a}_4^+\right]|p, \chi_2, \chi_3, \chi_4\rangle,
\end{aligned} \tag{23}$$

$$\begin{aligned}
\mathrm{i}\frac{\partial}{\partial\chi_4}\,|p, \chi_2, \chi_3, \chi_4\rangle &= \left[\left(-\frac{3}{4} + \mu_4\right)p - \mathrm{i}\frac{1}{4}\chi_2 - \mathrm{i}\frac{1}{2}\chi_3 - \mathrm{i}\frac{3}{4}\chi_4 + \mathrm{i}\frac{\sqrt{2}}{4}\hat{a}_1^+ \right. \\
&\quad \left. + \mathrm{i}\frac{\sqrt{2}}{4}\hat{a}_2^+ + \mathrm{i}\frac{\sqrt{2}}{4}\hat{a}_3^+ - \mathrm{i}\frac{3\sqrt{2}}{4}\hat{a}_4^+\right]|p, \chi_2, \chi_3, \chi_4\rangle.
\end{aligned} \tag{24}$$

Combining (19)–(24) we find that

$$\langle p,\chi_{2,}\chi_3,\chi_4|\hat{P}_2=-\mathrm{i}\frac{\partial}{\partial\chi_2}\langle p,\chi_{2,}\chi_3,\chi_4|, \tag{25}$$

$$\langle p,\chi_{2,}\chi_3,\chi_4|\hat{P}_3=-\mathrm{i}\frac{\partial}{\partial\chi_3}\langle p,\chi_{2,}\chi_3,\chi_4|, \tag{26}$$

$$\langle p,\chi_{2,}\chi_3,\chi_4|\hat{P}_4=-\mathrm{i}\frac{\partial}{\partial\chi_4}\langle p,\chi_{2,}\chi_3,\chi_4|. \tag{27}$$

5 Simultaneously Squeezing Unitary Transformation for $\hat{P}_1$, $\hat{X}_2$, $\hat{X}_3$ and $\hat{X}_4$

Since $[\hat{P}_1,\hat{X}_2]=[\hat{P}_1,\hat{X}_3]=[\hat{P}_1,\hat{X}_4]=0$, it is possible derive the squeezing operator for $\hat{P}_1$, $\hat{X}_2$, $\hat{X}_3$ and $\hat{X}_4$. We construct the following integral form unitary operator

$$U=\iiiint_{-\infty}^{\infty}dpd\chi_2d\chi_3d\chi_4\sqrt{\lambda\tau\sigma\nu}\,|\lambda p,\tau\chi_2,\sigma\chi_3,\nu\chi_4\rangle\,\langle p,\chi_2,\chi_3,\chi_4|\,, \tag{28}$$

where λ, τ, σ and ν are four independent positive numbers. The unitarity of U can be seen from (18) by calculating

$$\begin{aligned}
UU^{\dagger}&=\lambda\tau\sigma\nu\int\cdots\int_{-\infty}^{\infty}dpd\chi_2d\chi_3d\chi_4dp'd\chi_2'd\chi_3'd\chi_4'\,|\lambda p,\tau\chi_2,\sigma\chi_3,\nu\chi_4\rangle\,\langle p,\chi_2,\chi_3,\chi_4|\\
&\quad\times|p',\chi_2',\chi_3',\chi_4'\rangle\langle\lambda p',\tau\chi_2',\sigma\chi_3',\nu\chi_4'|\\
&=\lambda\tau\sigma\nu\int\cdots\int_{-\infty}^{\infty}dpd\chi_2d\chi_3d\chi_4dp'd\chi_2'd\chi_3'd\chi_4'\,|\lambda p,\tau\chi_2,\sigma\chi_3,\nu\chi_4\rangle\\
&\quad\times\langle\lambda p',\tau\chi_2',\sigma\chi_3',\nu\chi_4'|\delta(p'-p)\delta(\chi_2'-\chi_2)\delta(\chi_3'-\chi_3)\delta(\chi_4'-\chi_4)\\
&=\lambda\tau\sigma\nu\iiiint_{-\infty}^{\infty}dpd\chi_2d\chi_3d\chi_4\,|\lambda p,\tau\chi_2,\sigma\chi_3,\nu\chi_4\rangle\,\langle\lambda p,\tau\chi_2,\sigma\chi_3,\nu\chi_4|=1.
\end{aligned} \tag{29}$$

Using (8)–(11) and the IWOP technique, we can obtain

$$\begin{aligned}
U\hat{P}_1U^{\dagger}&=\lambda\tau\sigma\nu\int\cdots\int_{-\infty}^{\infty}dpd\chi_2d\chi_3d\chi_4dp'd\chi_2'd\chi_3'd\chi_4'|\lambda p,\tau\chi_2,\sigma\chi_3,\nu\chi_4\rangle\langle p,\chi_2,\chi_3,\chi_4|\\
&\quad\times\hat{P}_1|p',\chi_2',\chi_3',\chi_4'\rangle\langle\lambda p',\tau\chi_2',\sigma\chi_3',\nu\chi_4'|\\
&=\lambda\tau\sigma\nu\int\cdots\int_{-\infty}^{\infty}dpd\chi_2d\chi_3d\chi_4dp'd\chi_2'd\chi_3'd\chi_4'\,|\lambda p,\tau\chi_2,\sigma\chi_3,\nu\chi_4\rangle\,\langle p,\chi_2,\chi_3,\chi_4|\\
&\quad\times p'|p',\chi_2',\chi_3',\chi_4'\rangle\langle\lambda p',\tau\chi_2',\sigma\chi_3',\nu\chi_4'|\\
&=\lambda\tau\sigma\nu\int\cdots\int_{-\infty}^{\infty}dpd\chi_2d\chi_3d\chi_4dp'd\chi_2'd\chi_3'd\chi_4'\,|\lambda p,\tau\chi_2,\sigma\chi_3,\nu\chi_4\rangle\\
&\quad\times p'\delta(p'-p)\delta(\chi_2'-\chi_2)\delta(\chi_3'-\chi_3)\delta(\chi_4'-\chi_4)\langle\lambda p',\tau\chi_2',\sigma\chi_3',\nu\chi_4'|\\
&=\lambda\tau\sigma\nu\iiiint_{-\infty}^{\infty}pdpd\chi_2d\chi_3d\chi_4\,|\lambda p,\tau\chi_2,\sigma\chi_3,\nu\chi_4\rangle\,\langle\lambda p,\tau\chi_2,\sigma\chi_3,\nu\chi_4|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi^{-2}}{4}\lambda\tau\sigma\nu \iiiint_{-\infty}^{\infty} p dp d\chi_2 d\chi_3 d\chi_4 : \exp\bigg\{-\frac{\lambda^2}{4}p^2 - \frac{3\tau^2}{4}\chi_2^2 - \sigma^2\chi_3^2 - \frac{3\nu^2}{4}\chi_4^2 \\
&\quad - \tau\sigma\chi_2\chi_3 - \frac{\tau\nu}{2}\chi_2\chi_4 - \sigma\nu\chi_3\chi_4 \\
&\quad + \frac{\sqrt{2}}{4}\mathrm{i}\lambda p(\hat{a}_1^+ - \hat{a}_1 + \hat{a}_2^+ - \hat{a}_2 + \hat{a}_3^+ - \hat{a}_3 + \hat{a}_4^+ - \hat{a}_4) \\
&\quad + \frac{\sqrt{2}}{4}\tau\chi_2(3\hat{a}_1^+ + 3\hat{a}_1 - \hat{a}_2^+ - \hat{a}_2 - \hat{a}_3^+ - \hat{a}_3 - \hat{a}_4^+ - \hat{a}_4) \\
&\quad + \frac{\sqrt{2}}{4}\sigma\chi_3(2\hat{a}_1^+ + 2\hat{a}_1 + 2\hat{a}_2^+ + 2\hat{a}_2 - 2\hat{a}_3^+ - 2\hat{a}_3 - 2\hat{a}_4^+ - 2\hat{a}_4) \\
&\quad + \frac{\sqrt{2}}{4}\nu\chi_4(\hat{a}_1^+ + \hat{a}_1 + \hat{a}_2^+ + \hat{a}_2 + \hat{a}_3^+ + \hat{a}_3 - 3\hat{a}_4^+ - 3\hat{a}_4 \\
&\quad - \frac{1}{4}\sum_{i=1}^{4}(\hat{a}_i^{+^2} + \hat{a}_i^2) + \frac{1}{2}\sum_{j>i=1}^{j=N,i=N-1}(\hat{a}_i^+\hat{a}_j^+ + \hat{a}_i\hat{a}_j) - \sum_{i=1}^{4}\hat{a}_i^+\hat{a}_i\bigg\} : \\
&= \lambda^{-1}\frac{1}{\mathrm{i}\sqrt{2}}(\hat{a}_1 - \hat{a}_1^+ + \hat{a}_2 - \hat{a}_2^+ + \hat{a}_3 - \hat{a}_3^+ + \hat{a}_4 - \hat{a}_4^+) = \lambda^{-1}\hat{P}_1,
\end{aligned} \tag{30}
$$

where we have used the mathematical formula $\int_{-\infty}^{\infty} x e^{-\alpha x^2+\beta x}dx = \frac{\beta}{2\alpha}\sqrt{\frac{\pi}{\alpha}}e^{\frac{\beta^2}{4\alpha}}$, $\mathrm{Re}(\alpha) > 0$.

Similarly, we have

$$
U\hat{X}_2U^\dagger = \tau^{-1}\hat{X}_2, \qquad U\hat{X}_3U^\dagger = \sigma^{-1}\hat{X}_3, \qquad U\hat{X}_4U^\dagger = \nu^{-1}\hat{X}_4. \tag{31}
$$

Comparing (30) and (31), we see that U is indeed an operator simultaneously squeezing $\hat{P}_1$, $\hat{X}_2$, $\hat{X}_3$ and $\hat{X}_4$.

In summary, we have found a representation $|p, \chi_2, \chi_3, \chi_4\rangle$ in four-mode Fock space, which possesses orthonormal and completeness relation, and provide a new approach for solving some dynamic problems of four-body systems. The IWOP technique plays an essential role in our derivations.

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